Error analysis for the Millikan oil-drop experiment

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The electronic charge is derived from the measured fall time of a charged oil drop under the influence of gravitiy and the subsequent rise when an electric field is applied.

Ignoring the Cunningham correction factor f_c, the charge of an oil drop is obtained from the formula:

$$Q = \frac{6\pi\eta a}{E}\left(v_E + (-v_g)\right) \tag{1}$$

with η the viscosity of air, a the radius of the oil drop, E the electric field strenght, v_E the terminal velocity of the drop in the presence of an electric field and v_g the (terminal) velocity of the drop in the absense of the electric field. Remember that v_g is negative!

The drop radius can be found from the terminal velocity without electric field,

$$a = \sqrt{\frac{9\eta}{2g}\frac{(-v_g)}{\rho_{oil}}}. \tag{2}$$

Inserting this in eq. (1) yields the expression

$$Q = \frac{9\sqrt{2}\pi\eta^{3/2}}{E\sqrt{g\rho_{oil}}}\sqrt{(-v_g)}\left(v_E + (-v_g)\right). \tag{3}$$

Here, g is the gravitational constant. Since the velocity is measured as the time needed to travel a fixed distance, $|v| = \frac{d}{t}$, and the electric field is determined from the voltage V on two plates a distrance D apart, $E = V/D$, this can be written as

$$Q = \frac{9\sqrt{2}D\pi\eta^{3/2}}{V\sqrt{g\rho_{oil}}}\, d^{3/2}\sqrt{\frac{1}{t_g}}\left(\frac{1}{t_E} + \frac{1}{t_g}\right) \equiv K\sqrt{\frac{1}{t_g}}\left(\frac{1}{t_E} + \frac{1}{t_g}\right). \tag{4}$$

At this point, the separation between *common* (or *systematic*) and *independent* (or *statistical*) measurement errors is quite natural. The uncertainties in D, η, g and ρ_{oil} are common to all measurements. The uncertainties in t_g and t_E are independent and those in V and d need a little discussion.

The nature of the error in V depends on the way the device works. According to the specs, the voltage is 500 ± 1 V. If the voltage is *reproducable* in an on-off-on cycle to within 1 V, then the error in V must be treated as independent, since repeating the on-off-on

cycle many times will yield an *average* voltage which is almost *precisely* 500 V. However, if it reproduces to a much better precision, but the *absolute scale* is only known to within 1 V, then it must be considered a common error.

The same argument can be made for the uncertainty in d. If the uncertainty in d is taken to be the calibration of the microscope system, then the error must be treated as common, whereas if it is taken as the error with which the traveled distance can be know for each individual measurement, it must be considered independent. In the latter case the uncertainty in the time measurement should be taken much smaller to avoid double-counting.

For the following discussion, it is assumed that both uncertainties in V and d are common. In this case it is advantageous to rewrite eq. (4) as

$$Z \equiv \frac{Q}{K} = \sqrt{\frac{1}{t_g} \left(\frac{1}{t_E} + \frac{1}{t_g} \right)}. \tag{5}$$

The *statistical* uncertainty in Z, which arrises from the independent measurement errors, can be obtained straightforwardly

$$\Delta Z = \sqrt{\left(\frac{\partial Z}{\partial t_g} \right)^2 (\Delta t_g)^2 + \left(\frac{\partial Z}{\partial t_E} \right)^2 (\Delta t_E)^2}, \tag{6}$$

where it was assumed that the uncertainties in t_g and t_E are independent, *i.e.* that the measurement of t_g doesn't affect the measurement of t_E and *v.v.*. The uncertainties in t_g and t_E can be estimated based on the properties of the experimental setup or obtained experimentally by making multiple measurements, as described below.

If the transit time is measured multiple times, the *mean* can be calculated as

$$\bar{t} = \frac{1}{N} \sum_{i=1}^{N} t_i. \tag{7}$$

The spread in the data is usually characterised using the root-mean-squared (RMS) deviation of the data from its mean, which is defined as

$$s_t = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (t_i - \bar{t})^2}. \tag{8}$$

The *error on the mean*, $\Delta \bar{t}$ is then given by

$$\Delta \bar{t} = \frac{s_t}{\sqrt{N}} \tag{9}$$

The expressions for Z and ΔZ then become

$$\bar{Z} = \sqrt{\frac{1}{\bar{t}_g} \left(\frac{1}{\bar{t}_E} + \frac{1}{\bar{t}_g} \right)}. \tag{10}$$

and

$$\Delta \bar{Z} = \sqrt{\left(\frac{\partial Z}{\partial t_g}\right)^2 (\Delta \bar{t}_g)^2 + \left(\frac{\partial Z}{\partial t_E}\right)^2 (\Delta \bar{t}_E)^2} \tag{11}$$

When repeating the experiment several times for differently charged oil-drops ($Q = n\,e$), a set of discrete values will be measured for Z. Assuming that an accurate guess can be made for n in each measurement, the *normalized* value of Z, ϵ, can be obtained,

$$\epsilon_j \pm \Delta \epsilon_j \equiv \frac{\bar{Z}_j \pm \Delta \bar{Z}_j}{n_j}. \tag{12}$$

Since the uncertainties on all measurements of ϵ are independent, the *weighted average* may be calculated as

$$\bar{\epsilon} \equiv \frac{\sum\limits_{j=1}^{M} w_j \epsilon_j}{\sum\limits_{j=1}^{M} w_j} \tag{13}$$

with $w_j = (1/\Delta \epsilon_j)^2$. The uncertainty in the mean value is simply given by

$$\Delta \bar{\epsilon} = \frac{1}{\sqrt{\sum\limits_{j=1}^{M} w_j}} \tag{14}$$

Once all measurements are averaged, the final value of the electronic change can be calculated using eq. (4) and $Q = n\,e$,

$$\epsilon = \frac{Z}{n} = \frac{Q/n}{K} = \frac{e}{K}, \tag{15}$$

so that

$$e = K\,\epsilon \tag{16}$$

$$\Delta e = \sqrt{\left(\frac{\partial e}{\partial \epsilon}\right)^2 (\Delta \epsilon)^2 + \left(\frac{\partial e}{\partial K}\right)^2 (\Delta K)^2} = \sqrt{(\epsilon \Delta K)^2 + (K \Delta \epsilon)^2} \tag{17}$$

Since

$$K = \frac{9\sqrt{2} D \pi \eta^{3/2} d^{3/2}}{V \sqrt{g \rho_{oil}}}, \tag{18}$$

the uncertainty ΔK can be obtained as

$$\Delta K = \sqrt{\left(\frac{\partial K}{\partial D}\right)^2 (\Delta D)^2 + \left(\frac{\partial K}{\partial d}\right)^2 (\Delta d)^2 + \left(\frac{\partial K}{\partial V}\right)^2 (\Delta V)^2 + \cdots}$$

$$= K\sqrt{\left(\frac{\Delta D}{D}\right)^2 + \left(\frac{3\Delta d}{2d}\right)^2 + \left(\frac{\Delta V}{V}\right)^2 + \cdots} \tag{19}$$